

Aharonov-Bohm magnetism and Landau diamagnetism in semimetals

Eugene B. Kolomeisky¹ and Joseph P. Straley²

¹*Department of Physics, University of Virginia, P. O. Box 400714, Charlottesville, Virginia 22904-4714, USA*

²*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506-0055, USA*

We compute the magnetic response of hollow semimetal cylinders and rings to the presence of an axial Aharonov-Bohm magnetic flux, in the absence of interactions. We predict nullification of the Aharonov-Bohm effect for a class of dispersion laws that includes "non-relativistic" dispersion and demonstrate that at zero flux the ground-state of a very short "armchair" graphene tube will exhibit a ferromagnetic broken symmetry. We also compute the diamagnetic response of bulk semimetals to the presence of a uniform magnetic field, specifically predicting that the susceptibility has a logarithmic dependence on the size of the sample.

PACS numbers: 72.80.Vp, 73.23.-b, 73.23.Ad, 73.63.Fg

Ajiki and Ando (AA) [1] and Kane and Mele [2] have observed that the long-wavelength low-energy dynamics of the electrons of graphene [3] when confined to a cylindrical surface is described by the two-dimensional massless Dirac equation, and that the effects of the tube size and its chirality can be represented by a fictitious vector potential. If additionally there is an axial Aharonov-Bohm (AB) flux Φ present, then the energy eigenvalues of the Dirac equation are given by [1]

$$E_n(q_z) = \pm \gamma \left[q_z^2 + \left(\frac{2\pi}{W} \right)^2 (n + \phi \pm \alpha)^2 \right]^{\frac{1}{2}} \quad (1)$$

where the overall upper and lower signs refer to the conduction and valence bands (respectively), γ is the Fermi velocity v_F times \hbar , q_z is the wave vector in the axial direction, W is the cylinder circumference, $n = 0, \pm 1, \dots$ is the azimuthal quantum number, and $\phi = \Phi/\Phi_0$ is the dimensionless AB flux measured in units of the flux quantum $\Phi_0 = hc/e$. The fictitious flux parameter α combines the effects of winding and curvature, and provides a classification of nanotubes [1, 2]. It has opposite signs in the K and K' Dirac valleys [1] as indicated by the lower and upper signs in front of α in Eq.(1). This flux is not due to a physical magnetic field and for $\phi = 0$ the system is time-reversal symmetric. All nanotubes can be classified as being semimetallic "armchair" ($\alpha = 0$), insulating (α close to $1/3$), or semiconducting ($\alpha \ll 1$) [1, 2].

Since the integer part of α or ϕ can be absorbed into the definition of the azimuthal quantum number n , any physical property is a periodic function of α or ϕ with unit period. As an example of such a property, AA [1] calculated the AB magnetic response of an undoped cylinder, which is a conceptually interesting problem because at zero temperature the free carriers are absent and the effect is entirely due to the electrons of the filled Dirac sea. Additionally, graphene, while being a semimetal, represents a marginal case between normal metals (where AB magnetism is experimentally interpretable in terms of persistent currents [4]) and insulators (where the effect is expected to be suppressed due to the band gap). For a

non-chiral $\alpha = 0$ tube of length L , the valence electrons of given spin and belonging to the K -valley contribute to the ground-state energy the quantity

$$\mathcal{E}_K^{(cyl)}(\phi) = -\gamma \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L dq_z}{2\pi} \left[q_z^2 + \left(\frac{2\pi}{W} \right)^2 (n + \phi)^2 \right]^{\frac{1}{2}} \quad (2)$$

The corresponding magnetic moment $\mathcal{M}_K^{(cyl)}(\phi)$ and differential susceptibility $\chi_K^{(cyl)}(\phi)$ are determined by differentiation of Eq.(2):

$$\mathcal{M}_K^{(cyl)}(\phi) = -\frac{W^2}{4\pi\Phi_0} \frac{\partial \mathcal{E}_K^{(cyl)}}{\partial \phi}, \quad \chi_K^{(cyl)}(\phi) = \frac{W^2}{4\pi\Phi_0} \frac{\partial \mathcal{M}_K^{(cyl)}}{\partial \phi} \quad (3)$$

For a generic tube the ground-state energy (accounting for the spin degeneracy and including the contributions of both valleys) can be written in terms of $\mathcal{E}_K^{(cyl)}$ as

$$\mathcal{E}^{(cyl)}(\phi, \alpha) = 2(\mathcal{E}_K^{(cyl)}(\phi + \alpha) + \mathcal{E}_K^{(cyl)}(\phi - \alpha)), \quad (\mathcal{E} \rightarrow \mathcal{M} \rightarrow \chi) \quad (4)$$

and similar relationships hold for the total magnetic moment and susceptibility, with \mathcal{E} replaced by \mathcal{M} or χ . Thus by computing only one of the K -functions as a function of the flux ϕ we can understand the general problem.

In what follows we will be also interested in the one-dimensional (ring) version of the same problem that is obtained by setting $q_z \equiv 0$ in the spectrum Eq.(1). Then instead of Eq.(2) the ground-state energy is given by

$$\mathcal{E}_K^{(ring)}(\phi) = -\gamma \sum_{n=-\infty}^{\infty} \frac{2\pi}{W} |n + \phi| \quad (5)$$

and relationships analogous to those of Eq.(3) hold for the magnetic moment and susceptibility.

At low energies the "relativistic" dispersion law (1) emerges in a variety of physical systems [5], and so the problem of the AB magnetism of a semimetal goes beyond graphene. We will give a comprehensive treatment of the phenomenon by employing the zeta function regularization method [6] that finds wide applications in calculations of the Casimir effect. The flexibility and generality of the technique will allow us not only to solve

the above-mentioned "relativistic" versions of the problem but also, at no extra cost, discuss systems having more general dispersion laws. As a by-product we will also consider the Landau diamagnetism in semimetals.

Direct inspection of Eqs.(2) and (5) shows that they are divergent. The divergences are fictitious because the expression for the spectrum (1) is only applicable at low energy; furthermore, the sum and integral should only be over wavevectors within the first Brillouin zone. AA [1] treated this problem by introducing a cutoff function into Eq.(2) which allowed them to carry out a numerical calculation of the magnetic moment. From this they identified a cutoff-independent part which they argued captured the low energy part (1) of the true spectrum. A compact derivation of AA's result will be given below as a special case of a more general theory.

Our calculation follows the analysis of a similar problem [7]. We begin by defining the spectral zeta functions for the cylinder

$$\zeta_M^{(cyl)}(s) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \left[q_z^2 + \left(\frac{2\pi}{W} \right)^2 (n + \phi)^2 + M^2 \right]^{-\frac{s}{2}} \quad (6)$$

and the ring

$$\zeta_M^{(ring)}(s) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{2\pi}{W} \right)^2 (n + \phi)^2 + M^2 \right]^{-\frac{s}{2}} \quad (7)$$

versions of the problem. Here M confers a gap to the fermion spectrum, which will be put to zero later, and s is a parameter. For M finite and s positive and sufficiently large the expressions (6) and (7) are convergent and can be explicitly evaluated. The outcome will be analytically continued to the physically relevant situation of $M = 0$ and $s = -1$. This procedure extracts a cutoff-independent AB piece of the energies (2) and (5) via $\mathcal{E}_K^{(cyl)}(\phi) = -\gamma L \zeta_0^{(cyl)}(-1)$ and $\mathcal{E}_K^{(ring)}(\phi) = -\gamma \zeta_0^{(ring)}(-1)$ which is all we need to compute magnetic properties (3). The contribution that is dropped represents the cutoff-dependent ground-state energy of the filled Dirac sea in the absence of the AB flux; its value can be obtained by replacing the summation in Eqs.(2) and (5) by an integration.

The spectral zeta functions (6) and (7) contain the solutions to other problems, too. Forty years ago Abrikosov and Beneslavskii [8] demonstrated that (in three dimensions) crystal symmetry permits both linear $s = -1$ (like in graphene) and parabolic ("non-relativistic") $s = -2$ touching of the valence and conduction bands. The latter parallels parabolic dispersion law found in unbiased bilayer graphene [9] if the interactions are neglected; more generally, $s = -\nu$ describes a rhombohedral multilayer composed of ν graphene monolayers [3, 5]. Abrikosov and Beneslavskii additionally investigated the role of Coulomb interactions whose effect, like in graphene [10],

was shown to be fairly weak in the case of a linear spectrum. However, Coulomb interactions have a dramatic consequence for the case of a parabolic spectrum, $s = -2$, where a breakdown of single-particle description was predicted [8]. The situation in bilayer graphene is similar where recent experimental and theoretical work [11] found that interactions can lead to a reconstruction of the ground state. Additionally, the $s = -2$ case warrants special attention, because (i) the parabolic dispersion plays an important role in an explanation of the unconventional quantum Hall effect [9] and universal conductivity [12] in bilayer graphene, and (ii) it separates the regimes where the density of states is non-singular (for $-s < 2$) vs. singular (for $-s > 2$).

The spectral zeta functions (6) and (7) can be calculated by using the identity [13]

$$\int_0^\infty \frac{\cos p x dx}{(x^2 + a^2)^{\frac{s}{2}}} = \sqrt{\pi} \left(\frac{p}{2a} \right)^{\frac{s-1}{2}} \frac{K_{\frac{s-1}{2}}(pa)}{\Gamma(\frac{s}{2})}, \quad \Re s > 0 \quad (8)$$

where $\Gamma(z)$ is the Gamma function and $K_\mu(z)$ is the (MacDonald) modified Bessel function. For $\Re s > 1$ this permits integration of (6) over q_z , leading to

$$\zeta_M^{(cyl)}(s) = \frac{\Gamma(\frac{s-1}{2})}{2\sqrt{\pi}\Gamma(\frac{s}{2})} \zeta_M^{(ring)}(s-1) \quad (9)$$

which relates the cylinder (6) and ring (7) spectral zeta functions. Thus knowledge of the full s dependence of one of them describes both cases [14]. For example, the AB magnetism of a graphene ring (described by $\zeta_{M=0}^{(ring)}(-1)$) can be inferred via (9) from the zeta function for the cylinder (as $\zeta_{M=0}^{(cyl)}(0)$).

When $\Re s > 1$, the spectral zeta function for the ring (7) can be computed by employing the Poisson summation formula. With the aid of (8) we find

$$\begin{aligned} \zeta_M^{(ring)}(s) &= \frac{4\sqrt{\pi}}{\Gamma(\frac{s}{2})} \left(\frac{W}{2\pi} \right)^s \\ &\times \left\{ \frac{\Gamma(\frac{s-1}{2})}{4} \left(\frac{MW}{2\pi} \right)^{1-s} + \left(\frac{2\pi^2}{MW} \right)^{\frac{s-1}{2}} \right. \\ &\times \left. \sum_{n=1}^{\infty} \frac{\cos 2\pi n \phi}{n^{\frac{1-s}{2}}} K_{\frac{s-1}{2}}(nMW) \right\} \quad (10) \end{aligned}$$

Combined with (9) this provides us with an expression for the cylinder spectral zeta function

$$\begin{aligned} \zeta_M^{(cyl)}(s) &= \frac{2}{\Gamma(\frac{s}{2})} \left(\frac{W}{2\pi} \right)^{s-1} \\ &\times \left\{ \frac{\Gamma(\frac{s-2}{2})}{4} \left(\frac{MW}{2\pi} \right)^{2-s} + \left(\frac{2\pi^2}{MW} \right)^{\frac{s-2}{2}} \right. \\ &\times \left. \sum_{n=1}^{\infty} \frac{\cos 2\pi n \phi}{n^{\frac{2-s}{2}}} K_{\frac{s-2}{2}}(nMW) \right\} \quad (11) \end{aligned}$$

valid for $\Re s > 2$. These expressions can be analytically continued into the $\Re s < 1$ and $\Re s < 2$ regions, respectively, and the $M \rightarrow 0$ limit can be taken which leads to our main results

$$\zeta_0^{(cyl)}(s) = \frac{\Gamma(1 - \frac{s}{2})}{\pi\Gamma(\frac{s}{2})} \left(\frac{2}{W}\right)^{1-s} \sum_{n=1}^{\infty} \frac{\cos 2\pi n\phi}{n^{2-s}} \quad (12)$$

$$\zeta_0^{(ring)}(s) = \frac{2\Gamma(\frac{1-s}{2})}{\sqrt{\pi}\Gamma(\frac{s}{2})} \left(\frac{2}{W}\right)^{-s} \sum_{n=1}^{\infty} \frac{\cos 2\pi n\phi}{n^{1-s}} \quad (13)$$

As a first application of Eq.(12) we consider the $s = -1$ case, which describes a cylinder with a linear dispersion law (1). The part of the ground-state energy that depends on the AB flux will be given by

$$\mathcal{E}_K^{(cyl)}(\phi) = -\gamma L \zeta_0^{(cyl)}(-1) = \frac{\gamma L}{\pi W^2} \sum_{n=1}^{\infty} \frac{\cos 2\pi n\phi}{n^3} \quad (14)$$

The AB flux controls both the magnitude and sign of the result; the energy has maxima at ϕ integer and minima at ϕ half-odd integer. The magnetic moment and susceptibility of the cylinder follow from Eqs.(3) as:

$$\mathcal{M}_K^{(cyl)}(\phi) = \frac{\gamma L}{2\pi\Phi_0} \sum_{n=1}^{\infty} \frac{\sin 2\pi n\phi}{n^2} \rightarrow -\frac{\gamma L}{\Phi_0} \int_0^\phi dt \ln 2 |\sin \pi t| \quad (15)$$

$$\chi_K^{(cyl)}(\phi) = \frac{\gamma W^2 L}{4\pi\Phi_0^2} \sum_{n=1}^{\infty} \frac{\cos 2\pi n\phi}{n} \rightarrow -\frac{\gamma W^2 L}{4\pi\Phi_0^2} \ln 2 |\sin \pi\phi| \quad (16)$$

where the last two representations, valid in the $0 < \phi < 1$ range, should be periodically continued for all other ϕ . We observe that both quantities are proportional to the cylinder length L ; the magnetic moment is independent of the circumference W , while the susceptibility is logarithmically divergent at ϕ integer. This translates into a weak non-analyticity at integer values of ϕ for the energy (14) and magnetic moment (15), with the latter vanishing both at integer and half-odd integer ϕ . Eqs.(15) and (16), combined with Eqs.(4) for the total magnetic moment and susceptibility, reproduce the AA results [1].

As an application of Eq.(13) we consider the $s = -1$ case which will describe a ring with linear dispersion law. The AB piece of the ground-state energy will be given by

$$\begin{aligned} \mathcal{E}_K^{(ring)}(\phi) &= -\gamma \zeta_0^{(ring)}(-1) = \frac{2\gamma}{\pi W} \sum_{n=1}^{\infty} \frac{\cos 2\pi n\phi}{n^2} \\ &\rightarrow \frac{2\pi\gamma}{W} \left(\frac{1}{6} - \phi + \phi^2\right) \end{aligned} \quad (17)$$

where the last representation is valid for the range $0 < \phi < 1$ and should be periodically continued for all other ϕ . As in the case of the cylinder, the magnitude and sign

can be controlled by the AB flux ϕ . The maxima of (17) are located at ϕ integer while the minima lie at half-odd integer ϕ . The magnetic moment then follows as

$$\mathcal{M}_K^{(ring)}(\phi) = \frac{\gamma W}{2\Phi_0} (1 - 2\phi), \quad 0 < \phi < 1 \quad (18)$$

It is proportional to the circumference W , varies between $\gamma W/2\Phi_0$ and $-\gamma W/2\Phi_0$, and has discontinuities at ϕ integer. The susceptibility is constant and diamagnetic, except for positive delta-function peaks at ϕ integer.

A remarkable feature of the ring geometry is that it displays a "ferromagnetic" broken symmetry at zero flux: the magnetic moment can be of either sign, depending on the history. This was already implied by Eq.(5), which has discontinuous derivative at every integer value of ϕ . A possible experimental realization of such a ring represents an "armchair" graphene cylinder ($\alpha = 0$ case of Eqs.(1) and (4)) whose length is much smaller than its circumference, $L \ll W$. Such cylinders are not yet experimentally available but hopefully peculiarity of their ground state would stimulate efforts to produce them.

The case of the parabolic dispersion law $s = -2$ holds a surprise: the cylinder and the ring spectral zeta functions (12) and (13) (as well as the corresponding magnetic moments) vanish at $s = -2$. In fact, this remains true for any even $\nu = -s$ because this is where the Gamma function in the denominators of Eqs.(12) and (13) have poles. We thus conclude that the AB effect does not exist for a cylinder or ring with an even-layer rhombohedral graphene wall. Inspection of Eqs.(10) and (11) shows that for even $\nu = -s$ the AB effect is also identically zero in the presence of a gap which covers the case of a dielectric or a filled band of a metal ($s = -2$, $M \neq 0$).

The AB magnetism is physically closely related to the Landau diamagnetism as the latter is also due to currents circulating along the surface of the sample [4]. The mathematical description of the two effects is also very similar. Indeed, for a linear dispersion law the energy eigenvalues of the three-dimensional Dirac equation are given by [15] (compare with (1))

$$E_n(q_z) = \pm \gamma \left(q_z^2 + \frac{2eH}{\hbar c} n \right)^{\frac{1}{2}}, \quad n = 0, 1, 2, \dots \quad (19)$$

where q_z is parallel to the magnetic field H . Each of the Landau levels (19) labeled by n has a degeneracy $eHA/2\pi\hbar c$, where A is the cross-sectional area of the sample perpendicular to the direction of the field. In a semimetal, all negative energy Landau levels are filled while the $n = 0$ state (shared between the valence and conduction bands) is half-filled. With this in mind the ground-state energy can be written as (compare with (2))

$$\mathcal{E}^{(3d)}(H) = -\frac{\gamma eHA}{4\pi\hbar c} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L dq_z}{2\pi} \left(q_z^2 + \frac{2eH}{\hbar c} |n| \right)^{\frac{1}{2}} \quad (20)$$

where L is the height of the sample in the direction of the magnetic field. For a two-dimensional semimetal in a magnetic field perpendicular to the plane of the sample we would instead write (compare with (5))

$$\mathcal{E}^{(2d)}(H) = -\frac{\gamma e H A}{4\pi\hbar c} \sum_{n=-\infty}^{\infty} \left(\frac{2eH}{\hbar c} |n| \right)^{\frac{1}{2}} \quad (21)$$

As in the AB case, let us define the spectral zeta functions (compare with (6) and (7) for $M = 0$)

$$\zeta_0^{(3d)}(s) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \left(q_z^2 + \frac{2eH}{\hbar c} |n| \right)^{-\frac{s}{2}} \quad (22)$$

$$\zeta_0^{(2d)}(s) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{2eH}{\hbar c} \right)^2 n^2 \right\}^{-\frac{s}{4}} \quad (23)$$

Comparing Eqs.(7) and (23) we notice that the latter can be analytically continued into the physically interesting region of negative s by setting $\phi = 0$ and replacing $2\pi/W \rightarrow 2eH/\hbar c$ and $s \rightarrow s/2$ in Eq.(13):

$$\zeta_0^{(2d)}(s) = \frac{2\Gamma(\frac{2-s}{4})}{\sqrt{\pi}\Gamma(\frac{s}{4})} \left(\frac{2eH}{\pi\hbar c} \right)^{-\frac{s}{2}} \zeta\left(\frac{2-s}{2}\right) \quad (24)$$

where $\zeta(z)$ is the Riemann zeta function. The part of the energy (21) per unit area dependent on the magnetic field is then given by

$$\frac{\mathcal{E}^{(2d)}(H)}{A} = -\frac{\gamma e H}{4\pi\hbar c} \zeta_0^{(2d)}(-1) = \frac{\gamma\zeta(\frac{3}{2})}{16\pi^2} \left(\frac{2eH}{\hbar c} \right)^{\frac{3}{2}} \quad (25)$$

Multiplied by 4 (graphene's degeneracy factor), this reproduces the result foreseen as early as 1989 [16].

Since the spectral functions (22) and (23) satisfy the relationship (9), the latter combined with (24) provides us with the analytic continuation of (22) into the region of physically interesting s :

$$\zeta_0^{(3d)}(s) = \frac{\Gamma(\frac{s-1}{2})\Gamma(\frac{3-s}{4})}{\pi\Gamma(\frac{s}{2})\Gamma(\frac{s-1}{4})} \left(\frac{2eH}{\pi\hbar c} \right)^{-\frac{s-1}{2}} \zeta\left(\frac{3-s}{2}\right) \quad (26)$$

The physically relevant case now holds a surprise because at $s = -1$ the spectral zeta function (26) has a pole:

$$\zeta_0^{(3d)}(s \rightarrow -1) \rightarrow -\frac{eH}{6\pi\hbar c} \frac{1}{s+1} \rightarrow -\frac{eH}{6\pi\hbar c} \ln \frac{L}{b} \quad (27)$$

This is a sign of a logarithmic cutoff dependence; the residue of the spectral function provides us with the amplitude of the logarithm [17] as indicated in the last step. Here b is of the order of the interparticle spacing. The magnetic piece of the energy (20) per unit volume is then given by

$$\frac{\mathcal{E}^{(3d)}(H)}{AL} = -\frac{\gamma e H}{4\pi\hbar c} \zeta_0^{(3d)}(-1) = \frac{v_F}{24\pi^2 c} \frac{e^2}{\hbar c} H^2 \ln \frac{L}{b} \quad (28)$$

where, to give a better idea of the magnitude of the effect, we substituted $\gamma = \hbar v_F$. The magnetization $\mathcal{M}^{(3d)}$ and magnetic susceptibility $\chi^{(3d)}$ then follow as

$$\mathcal{M}^{(3d)}(H) = \chi^{(3d)} H, \quad \chi^{(3d)} = -\frac{v_F}{12\pi^2 c} \frac{e^2}{\hbar c} \ln \frac{L}{b} \quad (29)$$

The last equation erroneously predicts that as $L \rightarrow \infty$, the susceptibility drops below the ideal diamagnetic limit of $-1/4\pi$. This means that a more careful treatment is needed that distinguishes between the external magnetic field H and the magnetic induction B representing the field experienced by the electrons of the substance [18]. This can be accomplished by replacing H with B in Eq.(29) which no longer gives $\mathcal{M}(H)$; the latter dependence can be found from the equation $H = B - 4\pi\mathcal{M}(B)$. As a result correct version of Eqs.(29) would read

$$\mathcal{M}(H) = \chi H, \quad \chi = \frac{\chi^{(3d)}}{1 - 4\pi\chi^{(3d)}} \quad (30)$$

where we assumed that a cylindrical sample is placed in an external axial magnetic field H . We now see that in the thermodynamic limit $L \rightarrow \infty$ the susceptibility χ approaches $-1/4\pi$, i.e. the bulk semimetal is an ideal diamagnet. In practice, however, we have $|\chi| \ll 1$ and Eqs.(29) are adequate, as the amplitude of the logarithm in (29) is of the order 10^{-6} ; astronomically large sample sizes would be required to observe $|\chi| \approx 1/4\pi$.

This work was supported by the US AFOSR.

-
- [1] H. Ajiki and T. Ando, J. Phys. Soc. Jpn. **62**, 1255 (1993); *ibid.* **62**, 2470 (1993); T. Ando, J. Phys. Soc. Jpn. **74**, 777 (2005), and references therein.
 - [2] C. L. Kane and E. J. Mele, Phys. Rev. Lett., **78**, 1932 (1997).
 - [3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009), and references therein.
 - [4] I. O. Kulik, Pis'ma Zh. Eksp. Teor. Fiz. **11**, 407 (1970) [Sov. Phys. JETP Lett. **11**, 275 (1970)]; in *Quantum Mesoscopic Phenomena and Mesoscopic Devices in Microelectronics* (Kluwer Academic Publishers, 2000), p. 259, and references therein; Low. Temp. Phys. **36**, 841 (2010), and references therein.
 - [5] T.T. Heikkilä, N.B. Kopnin and G.E. Volovik, Pisma v ZhETF, **94**, 252 (2011) [JETP Lett., **94**, 233(2011)], and references therein.
 - [6] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).
 - [7] E. N. Bogachev, I. V. Krive and A. S. Rozhanskii, Theor. Math. Phys. **83**, 419 (1990).
 - [8] A. A. Abrikosov and S. D. Beneslavskii, Zh. Eksp. Teor. Fiz. **59**, 1280 (1970) [Sov. Phys. JETP **32**, 699 (1971)]; J. Low Temp. Phys. **5**, 141 (1971).
 - [9] K. S. Novoselov, *et al.*, Nature Physics **2**, 177 (2006); E. McCann and V. I. Fal'ko, Phys. Rev. Lett. **96**, 086805 (2006).

- [10] D. C. Elias, *et al.*, Nature Physics **7**, 701 (2011); J. Gonzalez, F. Guinea, and M. A. H. Vozmediano, Nucl. Phys. B **424**, 595 (1994), Phys. Rev. B **59**, 2474 (1999); D. E. Sheehy and J. Schmalian, Phys. Rev. Lett. **99**, 226803 (2007); M. S. Foster and I. L. Aleiner, Phys. Rev. B **77**, 195413 (2008).
- [11] A. S. Mayorov, *et al.*, Science **333**, 860 (2011); O. Vafek and K. Yang, Phys. Rev. B **81**, 041401(R) (2010); F. Zhang, H. Min, M. Polini, and A. H. MacDonald, Phys. Rev. B **81**, 041402 (2010); R. Nandkishore and L. S. Levitov, Phys. Rev. Lett. **104**, 156803 (2010).
- [12] M. I. Katsnelson, Eur. Phys. J. B **52**, 151 (2006).
- [13] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, 5th ed. (Academic, New York, 1994), 3.771.2.
- [14] V. V. Nesterenko and I. G. Pirozhenko, J. Math. Phys. **41**, 4521 (2000).
- [15] A. A. Abrikosov, Phys. Rev. B **58**, 2788 (1998).
- [16] A. A. Nersesyan and G. E. Vachnadze, J. Low Temp. Phys. **77**, 293 (1989).
- [17] E. B. Kolomeisky, J.P. Straley, L. S. Langsjoen and H. Zaidi, J. Phys. A: Math. Theor. **43**, 385402 (2010); E. B. Kolomeisky, H. Zaidi, L. S. Langsjoen and J. P. Straley, arXiv:1110.0421v1.
- [18] A. A. Abrikosov, *Fundamentals of the Theory of Metals*, (North Holland, 1988), Section 10.6.